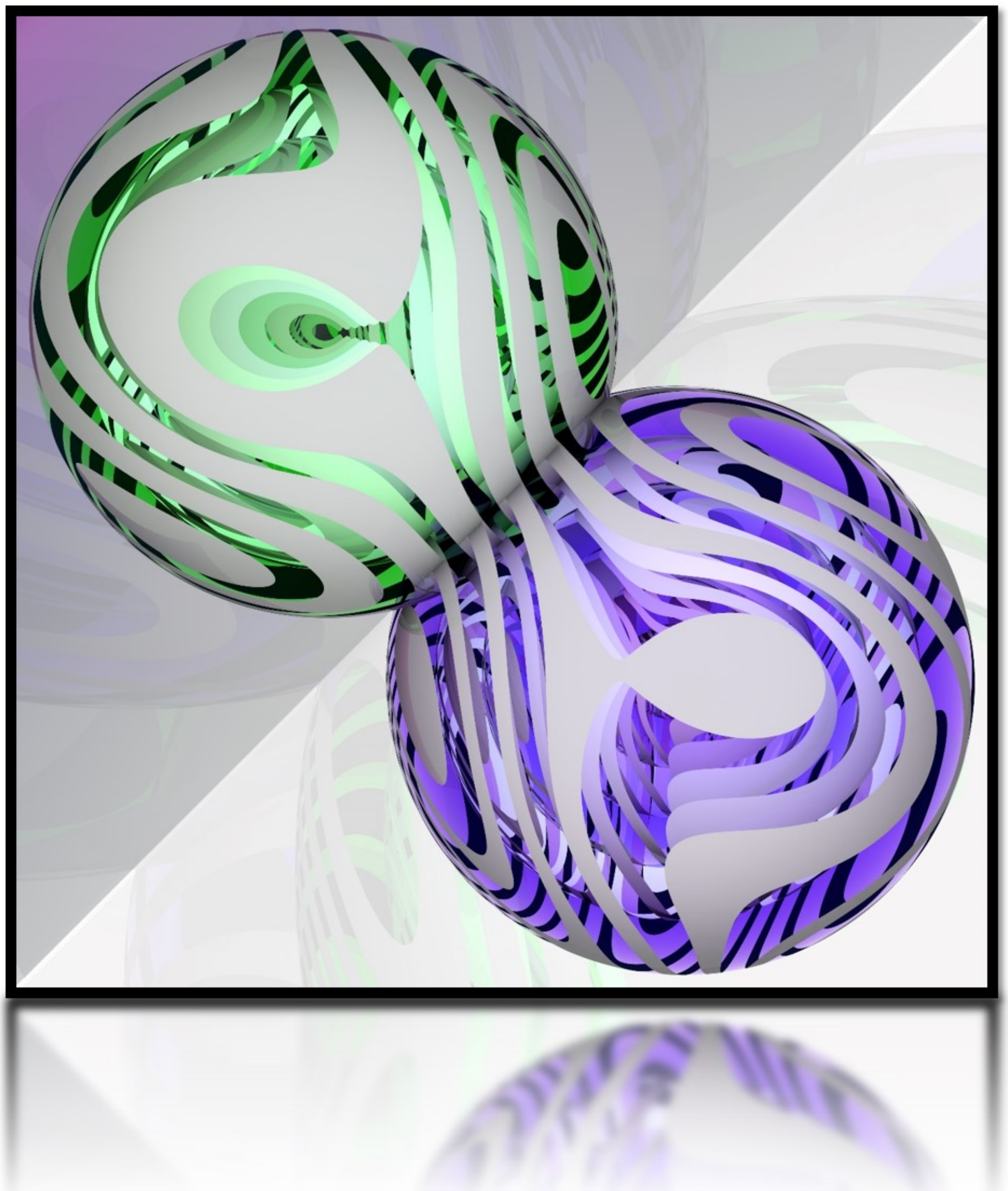
5. Cohomology



5.1 Cochain Complexes

The **dual** V^* of a $\mathbb{F}$-vector space consists of all linear maps $V \to \mathbb{F}$. It is not too painful to confirm that V^* is also a vector space over $\mathbb{F}$ — given linear maps p, q in V^* along with scalars α, β in $\mathbb{F}$, the linear combination $\alpha \cdot p + \beta \cdot q$ is evidently another linear map $V \to \mathbb{F}$ and hence constitutes an element of V^*. For finite-dimensional V one can describe the elements of V^* quite explicitly — every basis $\{e_1, \ldots, e_k\} \subset V$ has a corresponding *dual basis* $\{e_1^*, \ldots, e_k^*\} \subset V^*$ prescribed by the defined by the following action on the V-basis:

$$e_i^*(e_j) = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}.$$

Thus, we can transport any basis for V to a basis for V^* and express all the elements of V^* in terms of this dual basis.

Life gets considerably more interesting when one similarly attempts to dualize a linear map $A : V \to W$ of $\mathbb{F}$-vector spaces. Now A does not give us any straightforward way of sending V^*-elements to W^*-elements — every $p : V \to \mathbb{F}$ fits into an awkward zigzag with A:

$$W \xleftarrow{A} V \xrightarrow{p} \mathbb{F},$$

In sharp contrast, if we start with $q : W \to \mathbb{F}$, then there *is* an obvious map $V \to \mathbb{F}$:

$$V \xrightarrow{A} W \xrightarrow{q} \mathbb{F}.$$

Thus, for every $A : V \to W$ we get a **dual map** $A^* : W^* \to V^*$ which acts as $q \mapsto q \circ A$. Our goal here is to investigate some of the homological consequences of this dramatic reversal of domain and codomain that occurs when we dualize linear maps.

Let's start with a chain complex $(C_\bullet, d_\bullet)$ over $\mathbb{F}$

$$\cdots \xrightarrow{d_{k+1}} C_k \xrightarrow{d_k} C_{k-1} \xrightarrow{d_{k-1}} \cdots \xrightarrow{d_2} C_1 \xrightarrow{d_1} C_0 \longrightarrow 0$$

and dualize everything in sight:

$$\cdots \xleftarrow{d^*_{k+1}} C^*_k \xleftarrow{d^*_k} C^*_{k-1} \xleftarrow{d^*_{k-1}} \cdots \xleftarrow{d^*_2} C^*_1 \xleftarrow{d^*_1} C_0 \longleftarrow 0$$

The important fact from out perspective is that even in this dualized form, adjacent maps compose to give zero; given any dimension $k \geq 0$ and linear map $\zeta : C_k \to \mathbb{F}$, we have $d^*_{k+2} \circ d^*_{k+1}(\zeta) = \zeta \circ d_{k+1} \circ d_{k+2}$, which must equal zero regardless of ζ by the defining property of a chain complex. If we write this dualized chain complex from left to right and shift the indexing of the dual boundary maps by 1, then we arrive at the following definition.

Definition 5.1. A **cochain complex** $(C^\bullet, d^\bullet)$ over $\mathbb{F}$ is a sequence of vector spaces and linear maps of the form

$$0 \longrightarrow C^0 \xrightarrow{d^0} C^1 \xrightarrow{d^1} C^2 \xrightarrow{d^2} \cdots \xrightarrow{d^{k-1}} C^k \xrightarrow{d^k} C^{k+1} \xrightarrow{d^{k+1}} \cdots$$

satisfying $d^{k-1} \circ d^k = 0$ for every $k \geq 1$.

Aside from the fact that the maps are going up the indexing rather than down, cochain complexes are not very different from the chain complexes of Definition 3.9. We call C^k the k-th *cochain group* and $d^k : C^k \to C^{k+1}$ the k-th *coboundary map* of $(C^\bullet, d^\bullet)$.

5.2 Cohomology

Let $(C^\bullet, d^\bullet)$ be a cochain complex over a field $\mathbb{F}$.

Definition 5.2. For each dimension $k \geq 0$, the k-th **cohomology group** of $(C^\bullet, d^\bullet)$ is the quotient vector space

$$\mathbf{H}^k(C^\bullet, d^\bullet) = \ker d^k / \operatorname{img} d^{k-1}$$

Elements of $\ker d^k$ are called *k-cocycles* while elements of $\operatorname{img} d^{k-1}$ are the *k-coboundaries*. To acquire geometric intuition for cohomology, we will retreat to the relative comfort of simplicial complexes.

Let K be a simplicial complex, so that each chain group $\mathbf{C}_k(K)$ is generated by treating the k-simplices as basis elements. Thus, each k-simplex σ in K corresponds to a distinguished cochain $\sigma^* : \mathbf{C}_k(K) \to \mathbb{F}$, defined by (linearity and) the following action on any given k-simplex τ:

$$\sigma^*(\tau) = \begin{cases} 1 & \tau = \sigma \\ 0 & \tau \neq \sigma \end{cases}$$

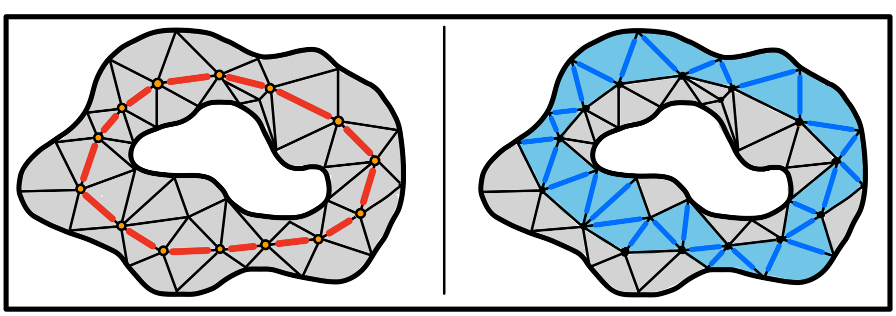
The collection of such cochains $\{\sigma^* : \mathbf{C}_k(K) \to \mathbb{F} \mid \dim(\sigma) = k\}$ forms a basis for the group of k-cochains of K. It is customary to write $\mathbf{C}^k(K)$ rather than the cumbersome $\mathbf{C}_k(K)^*$ to denote this *simplicial cochain group* of K — there is a long-standing convention in algebraic topology to index homology with subscripts and cohomology with superscripts.

The k-th *simplicial coboundary operator* is (unsurprisingly) denoted $\partial_K^k : \mathbf{C}^k(K) \to \mathbf{C}^{k+1}(K)$; by definition, this is the dual to the boundary operator $\partial_{k+1}^K : \mathbf{C}_{k+1}(K) \to \mathbf{C}_k(K)$, and hence satisfies $\partial_K^k(\sigma^*) = \sigma^* \circ \partial_{k+1}^K$ for each σ^* in $\mathbf{C}^k(K)$. It follows that for each general cochain ξ in $\mathbf{C}^k(K)$ and oriented $(k+1)$-simplex $\sigma = (v_0, \ldots, v_{k+1})$, we have the remarkably convenient formula

$$\partial_K^k \xi(\sigma) = \sum_{i=0}^{k} (-1)^i \cdot \xi(\sigma_{-i}), \tag{3}$$

where σ_{-i} is the face of σ obtained by deleting the vertex v_i. Thus, with respect to our choice of basis elements, ∂_K^k is simply the transpose[1] of the boundary matrix ∂_{k+1}^K for each $k \geq 0$; we will discuss three advantages of adopting this perspective shortly. In any event, the k-th cohomology group of the simplicial cochain complex $(\mathbf{C}^\bullet(K), \partial_K^\bullet)$ is called the k-th **simplicial cohomology group** of K and denoted by the shorthand $\mathbf{H}^k(K; \mathbb{F})$ or simply $\mathbf{H}^k(K)$.

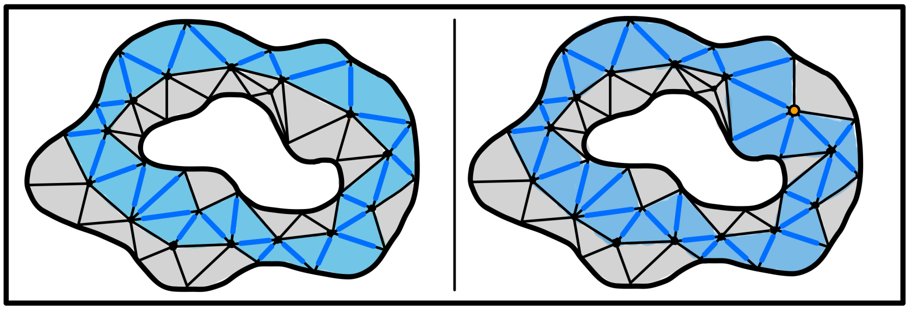
The first advantage of realizing that coboundary operators are transposes of boundary operators (with respect to our simplex-induced basis) is the ability to visualize low-dimensional simplicial cocycles at least over $\mathbb{F} = \mathbb{Z}/2$:



[1] When working with $\mathbb{F} = \mathbb{C}$ coefficients, this becomes a conjugate transpose.

To the left is a 1-cycle γ in a triangulated annulus, which we last saw when studying homology in Definition 3.11; and to the right we have a 1-cocycle ξ in the same annulus. All the edges being sent to 1 by ξ have been highlighted. On the left, every vertex had to be the face of an even number of edges in γ (otherwise the boundary $\partial_1(\gamma)$ would be nonzero). On the right, every 2-simplex must contain an even number of edges from ξ in its boundary (otherwise the coboundary $\partial^1(\xi)$ will be nonzero).

A second advantage is that we can also see in small examples when two cocycles lie in the same cohomology class; our 1-cocycle ξ represents the same cohomology class as new cocycle ξ' shown on the right, since they differ only by the coboundary of the highlighted vertex:



The third advantage of realizing that ∂^k_K is the transpose of ∂^K_{k+1} is the knowledge that they must have the same ranks as linear maps.

PROPOSITION 5.3. *Let* $(C_\bullet, d_\bullet)$ *be a chain complex over a field* $\mathbb{F}$ *so that* $\dim C_k$ *is finite for all* $k \geq 0$*, and let* $(C^\bullet, d^\bullet)$ *be its dual cochain complex. Then, we have*

$$\dim \mathbf{H}_k(C_\bullet, d_\bullet) = \dim \mathbf{H}^k(C^\bullet, d^\bullet)$$

in each dimension $k \geq 0$.

PROOF. This follows from the fact that $\dim C^k = \dim C_k$ and $\operatorname{rank} d^{k+1} = \operatorname{rank} d_k$ for all k:

$$\begin{aligned}
\dim \mathbf{H}_k(C_\bullet, d_\bullet) &= \dim\ker d_k - \dim \operatorname{img} d_{k+1} \\
&= (\dim C_k - \operatorname{rank} d_k) - \operatorname{rank} d_{k+1} \\
&= (\dim C^k - \operatorname{rank} d^{k-1}) - \operatorname{rank} d^k \\
&= (\dim C^k - \operatorname{rank} d^k) - \operatorname{rank} d^{k-1} \\
&= \dim\ker d^k - \dim \operatorname{img} d^{k-1} = \dim \mathbf{H}^k(C^\bullet, d^\bullet).
\end{aligned}$$

In particular, we have $\dim \mathbf{H}_k(K;\mathbb{F}) = \dim \mathbf{H}^k(K;\mathbb{F})$ for every simplicial complex K. ☐

The machinery developed for homology in the previous two chapters is readily translatable to work for cohomology, with the standard caveat that duality will force various maps to point in the opposite direction. For instance, every simplicial map $f : K \to L$ induces *cochain maps* $\mathbf{C}^\bullet f : (\mathbf{C}^\bullet(L), \partial^\bullet_L) \to (\mathbf{C}^\bullet(K), \partial^\bullet_K)$, which in turn yield well-defined linear maps

$$\mathbf{H}^k f : \mathbf{H}^k(L;\mathbb{F}) \to \mathbf{H}^k(K;\mathbb{F})$$

between cohomology groups. There is an avatar of Proposition 3.15 which allows us to extract bases of all the cohomology groups using Smith decompositions of coboundary matrices. Similarly, one can define cochain homotopies, relative cohomology groups and Mayer-Vietoris sequences for cohomology. This is a worthy endeavour, strongly recommended for all those who are encountering cohomology for the first time. Instead of reinventing that wheel here, we will focus on those aspects of cohomology which are new and different.

5.3 The Cup Product

The remarkable benefit of cochains over chains is that they are functions taking values in a field $\mathbb{F}$, so we can multiply them with each other. Fix an oriented simplicial complex K, so that each k-simplex σ can be uniquely written as an increasing list of vertices $(v_0, \ldots, v_k)$. It will be convenient henceforth to write, for each i in $\{0, \ldots, k\}$ the i-th *front face* of σ is the i-dimensional simplex $\sigma_{\leq i} = (v_0, \ldots, v_i)$, and similarly the i-th *back face* of σ is the $(k-i)$-dimensional simplex $\sigma_{\geq i} = (v_i, \ldots, v_k)$.

Definition 5.4. Let $\xi \in \mathbf{C}^k(K)$ and $\eta \in \mathbf{C}^\ell(K)$ be two simplicial cochains of K. Their **cup product** is a new cochain $\xi \smile \eta$ in $\mathbf{C}^{k+\ell}(K)$ defined by the following action on each $(k+\ell)$-dimensional simplex σ:

$$\xi \smile \eta(\sigma) = \xi(\sigma_{\leq k}) \cdot \eta(\sigma_{\geq k}).$$

(Here the multiplication on the right side takes place in the underlying field $\mathbb{F}$.)

Having obtained a new cochain $\xi \smile \eta$ by suitably multiplying ξ with η, we should lay to rest any curiosity regarding its coboundary.

Proposition 5.5. *For any ξ in $\mathbf{C}^k(K;\mathbb{F})$ and η in $\mathbf{C}^\ell(K;\mathbb{F})$, we have*

$$\partial_K^{k+\ell}(\xi \smile \eta) = [\partial_K^k(\xi) \smile \eta] + (-1)^k \cdot [\xi \smile \partial_K^\ell(\eta)].$$

Proof. Let τ be a $(k+\ell+1)$-dimensional oriented simplex with vertices $(v_0, \ldots, v_{k+\ell+1})$. We evaluate the two terms on the right side of the desired equality separately on τ. First,

$$\begin{aligned} [\partial_K^k(\xi) \smile \eta](\tau) &= \partial_K^k(\xi)(\tau_{\leq k+1}) \cdot \eta(\tau_{\geq k+1}) && \text{by Definition 5.4,} \\ &= \left(\sum_{i=0}^{k+1} (-1)^i \cdot \xi((\tau_{\leq k+1})_{-i}) \cdot \eta(\tau_{\geq k+1}) \right) && \text{by (3).} \end{aligned}$$

And similarly,

$$(-1)^k \cdot [\xi \smile \partial_K^\ell(\eta)](\tau) = \left(\sum_{j=0}^{\ell+1} (-1)^{k+j} \cdot \xi(\tau_{\leq k}) \cdot \eta((\tau_{\geq k})_{-j}) \right).$$

When we add these two expressions, the $i = k+1$ term of the first sum cancels the $j = 0$ term of the second; the terms which survive are exactly $\partial_K^{k+\ell}(\xi \smile \eta)(\tau)$ by (3). □

Using the above formula for the coboundary of $\xi \smile \eta$, one can confirm that the cup product of two cocycles is again a cocycle:

$$\begin{aligned} \partial_K^{k+\ell}(\xi \smile \eta) &= [\partial_K^k(\xi) \smile \eta] + (-1)^k[\xi \smile \partial_K^\ell(\eta)] && \text{by Proposition 5.5,} \\ &= [0 \smile \eta] + (-1)^k[\xi \smile 0] && \text{since } \xi \text{ and } \eta \text{ are cocycles,} \\ &= 0 && \text{by (3).} \end{aligned}$$

Now if $\xi = \partial_K^{k-1}(\xi')$ is a coboundary while and η is a cocycle as before, then their cup product is a coboundary:

$$\begin{aligned} \xi \smile \eta &= \partial_K^{k-1}(\xi') \smile \eta \\ &= [\partial^{k-1}(\xi') \smile \eta] + (-1)^k \cdot [\xi' \smile \partial_K^\ell(\eta)] && \text{since } \partial_K^\ell(\eta) = 0 \\ &= \partial_K^{k+\ell}(\xi' \smile \eta). && \text{by Proposition 5.5.} \end{aligned}$$

Similarly, if ξ is a cocycle and η a coboundary, then again their cup product is a coboundary. We have arrived at the following result.

PROPOSITION 5.6. *For each simplicial complex K and dimensions $k, \ell \geq 0$, the cup product map $\smile: \mathbf{C}^k(K;\mathbb{F}) \times \mathbf{C}^\ell(K;\mathbb{F}) \to \mathbf{C}^{k+\ell}(K;\mathbb{F})$ induces a well-defined bilinear map of cohomology groups.*

It is customary to use the same notation when describing the cup product on cohomology groups rather than cochains, i.e.,

$$\smile: \mathbf{H}^k(K;\mathbb{F}) \times \mathbf{H}^\ell(K;\mathbb{F}) \to \mathbf{H}^{k+\ell}(K;\mathbb{F}).$$

The direct sum $\bigoplus_{k\geq 0} \mathbf{H}^k(K;\mathbb{F})$ is evidently a vector space over $\mathbb{F}$; writing its elements as

$$\underline{\xi} = (\xi_1, \xi_2, \ldots, \xi_k, \ldots),$$

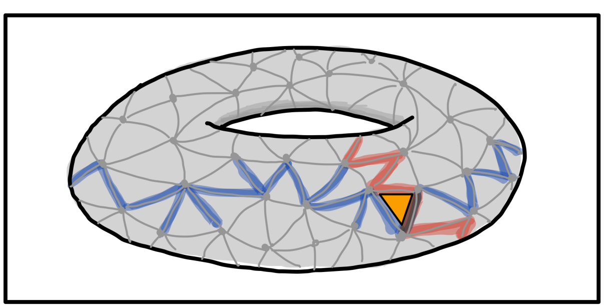
we say that $\underline{\xi}$ has *grade* k if all the ξ_i for $i \neq k$ are zero. The cup product gives us a bilinear multiplication law on this direct sum which is additive on grades, i.e., the cup product of a grade-k element with a grade-ℓ element is a grade-$(k+\ell)$ element. A graded $\mathbb{F}$-vector space equipped with such a graded bilinear multiplication is called a **graded algebra** over $\mathbb{F}$. While the direct sum of homology groups $\bigoplus_{k\geq 0} \mathbf{H}_k(K;\mathbb{F})$ also forms a graded vector space, there is no multiplication law analogous to the cup product. It is in this sense that cohomology is considered a richer algebraic invariant than homology, even though the dimensions of cohomology groups agree with those of homology groups when working over a field.

EXAMPLE 5.7. By a **torus** we mean any simplicial complex T whose geometric realization is homeomorphic to the product $\partial\Delta(2) \times \partial\Delta(2)$. Consider also the simplicial complex W obtained by first taking the disjoint union $\partial\Delta(3) \sqcup \partial\Delta(2) \sqcup \partial\Delta(2)$, and then identifying the vertices labelled $\{0\}$ of all three pieces to create a single connected simplicial complex. Now one can check that T and W have isomorphic homology groups over any field $\mathbb{F}$, namely

$$\mathbf{H}_k(T) = \mathbf{H}_k(W) = \begin{cases} \mathbb{F} & k \in \{0,2\} \\ \mathbb{F}^2 & k = 1 \\ 0 & k > 2 \end{cases}.$$

Let α and β denote any two 1-cycles which span $\mathbf{H}_1$ and examine their cup product $\alpha \smile \beta$. In T, this will be (a nonzero multiple of) the unique cycle generating $\mathbf{H}_2$, whereas in W this cup product will equal zero.

The cup product $\alpha \smile \beta$ in the torus is nontrivial for a viscerally geometric reason; one can choose α to be a cochain that runs along the equator while β runs along the meridian. Now there will be at least one 2-simplex whose 1-dimensional faces are *all* sent to nonzero elements of $\mathbb{F}$ by either α or β. We highlight such a 2-simplex for the illustrated α and β below:



The miracle here is that no matter how much we perturb α and β within their respective cohomology classes, we will always have at least one such 2-simplex.

REMARK 5.8. There are no obstacles to defining cohomology with non-field coefficients, e.g., by using coefficients sourced from the ring of integers $\mathbb{Z}$. However, various subtleties arise from the fact that in general an abelian group G is not isomorphic to its *dual group* G^*; here G^* consists of all abelian group homomorphisms $G \to \mathbb{Z}$. In particular, G^* is blind to torsion in G and there is no analogue of Proposition 5.3 when using $\mathbb{Z}$ coefficients. Similarly, in this case the cup product prescribes the structure of a **graded ring** on the direct sum $\bigoplus_{k\geq 0} \mathbf{H}^k(K;\mathbb{Z})$ rather than a graded algebra.

5.4 THE CAP PRODUCT

There is a second (far stranger) product that mixes homology and cohomology. As before, let K be an oriented simplicial complex; each oriented k-simplex σ therefore has a front face $\sigma_{\leq i}$ and a back face $\sigma_{\geq i}$ for i in $\{0,\ldots,k\}$. Our new product arises from taking an i-cochain ξ for some $i \leq k$ and letting it act on σ by

$$\sigma \mapsto \xi(\sigma_{\leq i}) \cdot \sigma_{\geq i}.$$

That is, we evaluate σ on the front face of the appropriate dimension, and multiply the resulting scalar with the back face to produce a chain of dimension $(k-i)$. More formally, note that each k-chain γ in $\mathbf{C}_k(K)$ is uniquely expressible as a linear combination $\gamma = \sum_\sigma \gamma_\sigma \cdot \sigma$ where σ ranges over oriented k-simplices and each γ_σ is an element of the coefficient field $\mathbb{F}$.

DEFINITION 5.9. The **cap product** of an i-cochain ξ with a k-chain $\gamma = \sum_\sigma \gamma_\sigma \cdot \sigma$ is the new $(k-i)$-chain $\xi \frown \gamma$ defined by

$$\xi \frown \gamma = \sum_\sigma \gamma_\sigma \cdot \xi(\sigma_{\leq i}) \cdot \sigma_{\geq i}.$$

(For $i > k$ this sum is automatically zero).

The first thing to confirm about the cap product formula from the definition above is that the expression on the right side is a $(k-i)$-chain — each $\sigma_{\geq i}$ is a $(k-i)$-simplex obtained by deleting the first i vertices of the k-simplex σ, and the product $\gamma_\sigma \cdot \xi(\sigma_{\leq i})$ of two $\mathbb{F}$-elements clearly lies in $\mathbb{F}$. By definition, the cap product gives us bilinear maps $\mathbf{C}^i(K) \times \mathbf{C}_k(K) \to \mathbf{C}_{k-i}(K)$ for every pair of dimensions $i \leq k$. Since $\xi \frown \gamma$ is a chain, it has a boundary rather than a coboundary.

PROPOSITION 5.10. *For each ξ in $\mathbf{C}^i(K)$ and γ in $\mathbf{C}_k(K)$, we have*

$$\partial^K_{k-i}(\xi \frown \gamma) = (-1)^i \cdot [(\xi \frown \partial^K_k(\gamma)) - (\partial^i_K(\eta) \frown \gamma)]$$

The above result follows from a calculation which has a very similar structure to the one which we used when proving Proposition 5.5. This has been assigned as an exercise, unlike the the following corollary.

PROPOSITION 5.11. *For each simplicial complex K and dimensions $i \leq k$, the cap product $\frown$: $\mathbf{C}^i(K;\mathbb{F}) \times \mathbf{C}_k(K;\mathbb{F}) \to \mathbf{C}_{k-i}(K;\mathbb{F})$ induces a well-defined bilinear map of cohomology groups.*

PROOF. The desired result follows from the three claims described below, each of which is proved using the boundary formula from Proposition 5.10.

1. cocycle $\frown$ cycle is a cycle: if $\partial^i_K(\xi) = 0$ and $\partial^K_k(\gamma) = 0$, then we get

$$\partial^K_{k-i}(\xi \frown \gamma) = (-1)^i \cdot [(\xi \frown 0) - (0 \frown \gamma)] = 0,$$

as desired.

2. cocycle $\frown$ boundary is a boundary: if $\partial^i_K(\xi) = 0$ and β is any $(k+1)$-chain, then by Proposition 5.10, we have

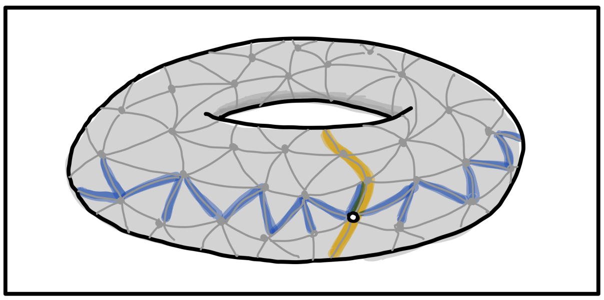
$$\pm(\xi \frown \partial^K_{k+1}(\beta)) = \partial^K_k(\xi \frown \beta) \mp (\partial^i_K(\eta) \frown \beta);$$

now the second term on the right side vanishes because η is a cocycle. Thus, up to a choice of sign, $\xi \frown \partial^K_{k+1}(\beta)$ equals $\partial^K_k(\xi \frown \beta)$ which is evidently a boundary.
3. coboundary $\frown$ cycle is a boundary: this is very similar to the previous claim, and has therefore been assigned as an exercise. ☐

As with the cup product, it is standard to use the same notation for the cap product map on (co)homology groups as on (co)chain groups:

$$\frown: \mathbf{H}^i(K; \mathbb{F}) \times \mathbf{H}_k(K; \mathbb{F}) \to \mathbf{H}_{k-i}(K; \mathbb{F}).$$

The geometry of the cap product is all about *intersections*. If we choose a meridinal 1-cycle γ and an equatorial 1-cocycle ξ on a torus as drawn below, then there will necessarily be at least one edge with a nonzero coefficient in γ that is sent to a nonzero element of $\mathbb{F}$ by ξ; and the 0-chain $\xi \frown \gamma$ will have a nonzero coefficient on one of the two vertices lying in the boudnary of that edge:



The power of the algebraic formulation of the cap product lies in the fact that the cycle $\xi \frown \gamma$ is well-defined on the level of homology even when ξ and γ are perturbed within their respective (co)homology classes.

5.5 Poincaré Duality

The cap product becomes extremely potent when applied to the study of *manifolds*. Throughout this section, we fix the following **assumption:**

> M is a simplicial complex whose geometric realization $|M|$ is a compact and connected n-dimensional manifold.

(The compactness requirement is overkill since we require simplicial complexes to be finite). The fact that every point on an n-manifold admits a local neighborhood homeomorphic to $\mathbb{R}^n$ forces every $(n-1)$-dimensional simplex of M to lie in the boundary of exactly two n-simplices.

Definition 5.12. We say that M is **orientable** over the field $\mathbb{F}$ if there exists a function

$$\omega : \{n-\text{simplices of } M\} \to \{\pm 1\}$$

assigning $\{\pm 1\} \subset \mathbb{F}$ to each top-dimensional simplex so that the chain $[M] = [M]_\omega$ given by

$$[M] = \sum_{\dim \sigma = n} \omega(\sigma) \cdot \sigma$$

is an n-cycle in $\mathbf{C}_n(M; \mathbb{F})$.

(It should be noted that any M satisfying our assumption above is automatically orientable in this sense over $\mathbb{F} = \mathbb{Z}/2$.) There is an unfortunate historical conflation of terminology here between orientability as defined above and the orderings of vertices which played a role in Definition 3.3. At any rate, if such a map ω exists then $[M]$ is called the **fundamental class** of M, and it generates all of $\mathbf{H}_n(M;\mathbb{F})$ which must necessarily be a one-dimensional vector space.

THEOREM 5.13. **[Poincaré duality.]** *Assume that M is a simplicial complex whose geometric realization is compact, connected and orientable over $\mathbb{F}$. For each i in $\{0,1,\ldots,n\}$, the linear map*

$$\boldsymbol{D}_i : \mathbf{H}^i(M;\mathbb{F}) \to \mathbf{H}_{n-i}(M;\mathbb{F})$$

given by $\boldsymbol{D}_i(\xi) = \xi \frown [M]$ is an isomorphism of $\mathbb{F}$-vector spaces.

It is quite challenging to prove this result entirely within the realm of simplicial complexes, so we will not make any such attempts here. But it should be noted that Poincaré duality has strong consequences for even the simplest homotopy invariants of manifolds. Combining Theorem 5.13 with Proposition 5.3 produces the following suite of results for Euler characteristics and Betti numbers of manifolds (see Sections 1 and 4 of Chapter 3).

COROLLARY 5.14. *Let M be a simplicial complex satisfying the hypotheses of Theorem 5.13. The following assertions hold.*

1. *The Betti numbers $\beta_0(M), \beta_1(M), \ldots, \beta_n(M)$ are palindromic, i.e., $\beta_k = \beta_{n-k}$ for all k.*
2. *If n is odd, then the Euler characteristic $\chi(M)$ is zero.*
3. *If $n = 2i$ is even, then the Euler characteristic $\chi(M;\mathbb{F})$ is odd if and only if the middle Betti number $\beta_i(M)$ is odd.*

PROOF. For the first assertion, note that

$$\begin{aligned}
\beta_k(M) &= \dim \mathbf{H}_k(\mathbb{F}) && \text{by definition,} \\
&= \dim \mathbf{H}^{n-k}(\mathbb{F}) && \text{by Theorem 5.13,} \\
&= \dim \mathbf{H}_{n-k}(\mathbb{F}) && \text{by Proposition 5.3,} \\
&= \beta_{n-k}(M) && \text{again by definition.}
\end{aligned}$$

The second assertion now follows from the first one by using (from Exercise 3.3) the fact that the Euler characteristic is the alternating sum of the Betti numbers:

$$\chi(M) = \sum_{k=0}^{n} (-1)^k \cdot \beta_k(M).$$

If n is odd, then β_k and β_{n-k} will appear with opposite signs and hence cancel. The third assertion follows from the same alternating sum — but for even $n = 2i$ all the β_k appear twice (with the same signs) except for β_i, which only appears once. Thus, the expression $\chi(M) \pm \beta_i(M)$ is always an even number. □

5.6 BONUS: THE KÜNNETH FORMULA

Let K and L be simplicial complexes. We have already lamented (in Section 8 of Chapter 4) that the product of simplicial complexes does not canonically have the structure of a simplicial complex. Even so, it is possible to find a simplicial complex P whose geometric realization is homeomorphic to $|K| \times |L|$, so it makes sense to define $\mathbf{H}_k(K \times L;\mathbb{F})$ as the usual homology groups of any such P, and similarly for cohomology groups. One naturally wonders how these product (co)homology groups of P relate to the (co)homology groups of the factors K and L.

Variants of the following result are called **Künneth formulas**.

THEOREM 5.15. *Let K and L be simplicial complexes and $\mathbb{F}$ a field. For each dimension $k \geq 0$ there is an isomorphism*

$$\mathbf{H}^k(K \times L; \mathbb{F}) \simeq \bigoplus_{i=0}^{k} \mathbf{H}^i(K; \mathbb{F}) \times \mathbf{H}^{k-i}(L; \mathbb{F}).$$

As a consequence of the Künneth formula and Proposition 5.3, one can compute Betti numbers of simplicial products via

$$\beta_k(K \times L) = \sum_{i=0}^{k} \beta_i(K) \cdot \beta_{k-i}(L).$$

There are several ways of proving Theorem 5.15; one strategy makes essential use of the cup product. Given a simplicial complex P whose realization is $|K| \times |L|$, assume that we have managed to simplicially approximate the natural projection maps from $K \times L$ to K and L, i.e.,

$$K \overset{f}{\twoheadleftarrow} P \overset{g}{\twoheadrightarrow} L$$

The goal now becomes to produce k-cochains in P from pairs of the form (ξ, η) where ξ is an i-cochain in K while η is a $(k-i)$-cochain in L. And the map which accomplishes this task is

$$(\xi, \eta) \mapsto \mathbf{C}^i f(\xi) \smile \mathbf{C}^{k-i} g(\eta).$$

EXERCISES

EXERCISE 5.1. Given a simplicial map $f : K \to L$, define the associated cochain maps $\mathbf{C}^k f : \mathbf{C}^k(L; \mathbb{F}) \to \mathbf{C}^k(K; \mathbb{F})$ and show that they commute with the coboundary operators (i.e., state and prove a cohomological version of Proposition 4.5).

EXERCISE 5.2. State a version of Definition 4.18 (short exact sequences) and Lemma 4.19 (the Snake lemma) that works for cochain complexes and cohomology.

EXERCISE 5.3. Show that the cup product is associative, i.e., for cochains ξ, η and ζ of a simplicial complex K, prove that

$$(\xi \smile \eta) \smile \zeta = \xi \smile (\eta \smile \zeta)$$

[Hint: by linearity, it suffices to evaluate both sides on a single simplex σ.]

EXERCISE 5.4. Let $f : K \to L$ be a simplicial map and consider a pair of cochains ξ in $\mathbf{C}^k(L)$ and η in $\mathbf{C}^\ell(L)$. Prove that $\mathbf{C}^k f(\xi) \smile \mathbf{C}^\ell f(\eta) = \mathbf{C}^{k+\ell} f(\xi \smile \eta)$. [Thus, we have $\mathbf{H}^k f(\xi) \smile \mathbf{H}^\ell f(\eta) = \mathbf{H}^{k+\ell}(\xi \smile \eta)$ whenever ξ and η lie in $\mathbf{H}^k(L)$ and $\mathbf{H}^\ell(L)$ respectively.]

EXERCISE 5.5. Prove Proposition 5.10.

EXERCISE 5.6. Prove the third claim of Proposition 5.11.

EXERCISE 5.7. Let $f : K \to L$ be a simplicial map. There is a diagram of $\mathbb{F}$-vector spaces, a part of which is shown below:

$$\begin{array}{ccccc} \mathbf{H}^i(K) & \times & \mathbf{H}_j(K) & \xrightarrow{\frown} & \mathbf{H}_{j-i}(K) \\ \\ \mathbf{H}^i(L) & \times & \mathbf{H}_j(L) & \xrightarrow{\frown} & \mathbf{H}_{j-i}(L) \end{array}$$

(1) draw three vertical arrows representing maps induced by f which connect the top row to the bottom row. What are the natural candidates for these maps?
(2) formulate an identity relating cap products and these three induced maps. You do not have to prove that this identity holds (but it is a good exercise to meditate on).

EXERCISE 5.8. Use the Künneth formula (Theorem 5.15) to find an expression for the k-th Betti number of the n-dimensional torus T^n obtained by taking the n-fold product of the hollow simplex $\partial\Delta(2)$.